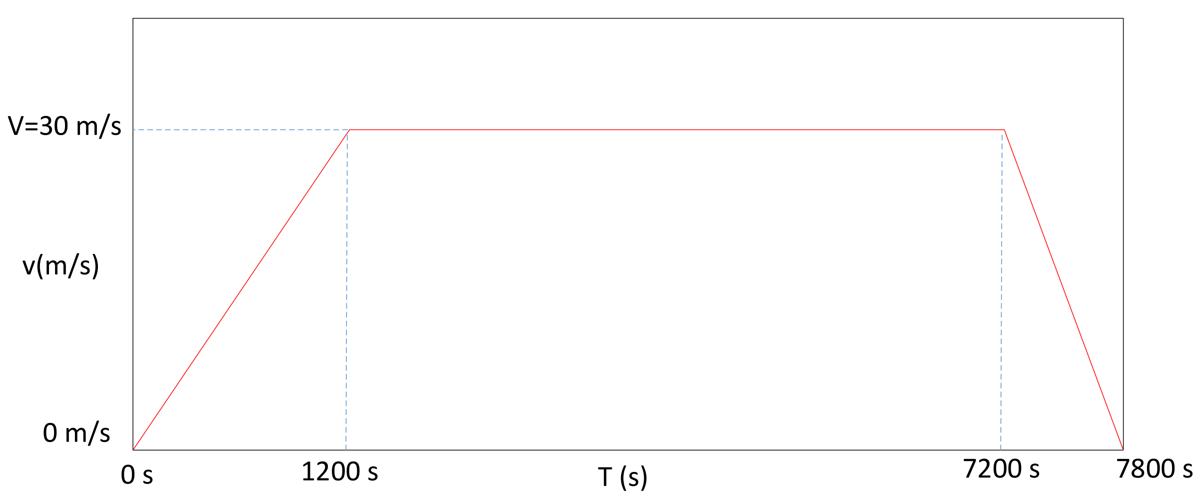
Motion is something we all take for granted. You get in your car and off you go to that job you so much enjoy. (Well maybe). That journey from house to work is bound to be a complex one. Travel down Main Street. Turn right on Fifth Street. Stop and go through the stop lights. Go around that turn. It is very complex indeed. Not only may your speed and direction along the North to South and East to West journey change, but your altitude as well (pending you don't live in Kansas). Over that hill there, to the next or down that mountain, so on and so on… Even so with all those changes it can be broken down into just three basic directions. Let north be up on the map and south down. East to the right and west to the left. Up out of the map towards your nose and down into the desk that map is sitting on. But let's simplify that even further. Let that North and South direction just be a variable call it y and let a positive y speed indicating you are going North and a negative y indicate you are going south. We just demoted two different directions into a single variable description. Now the same for East and West. Let that be x. Be it positive you are heading East my friend, and negative well you are heading toward California. That in and out of the paper that altitude. Let z represent that with a positive z you are moving up that mountain or hill and negative z down that mountain, foot on the break I hope. In this case we can denote at any 'exact' time your velocity that is your speed and direction. We just break it into those three components x, y and z. And what is your speed? It can be proven it is just the vector addition of those velocities $s^2=x^2+y^2+z^2$. 's' here being your speed. Speed states nothing about your direction and when the red and blue lights come on behind you. The man in the blue suit with pistol strapped on side cares not either. He just wants to know what your speed was. And your direction? Our friend trigonometry comes to the rescue: $a=atan(y/x)$. $u=sin(b)$. That is your angle, choose your units, it matters not to me. Degrees or Radians. 'a' being that angle from due east in this case. You tell your friend, I am heading 30 degrees north of due east that is. 'b' that is just the slope of that mountain. They always post if for the heavy laden trucks. He trucker dude gear down we have an 8 degree downhill for the next 3 miles it states. That is that trucks speed going 'down' well it's negative because we already choose that down be negative that is a negative z. 'u' now becomes a ratio multiply u by your speed s you get your downward speed component. Now your friend call's you on your cell phone and you do what everyone tells you not to, talk while driving on that phone. He asks you where you are going right now. You tell him your x, y, and z speed at this <u>moment</u> that is right at the point you are telling him. Does he have everything he needs to know? He has no idea what your speed and direction was just a moment ago or even what it will be a moment later, just like life, it is always changing. But if he could ascertain your velocity (that is those x, y, and z components) at every say 1/10 of a second of your journey to work. He would have enough information to determine where you were <u>at</u> during every moment of your travel. This is assuming that your velocity be not changing much from one tenth of a second to the next which it probably is not as massive objects such as cars take time to make any change in velocity or direction of travel. And how might he do that you ask? All he has to do is take each velocity at each moment, multiply it by one tenth of a second and he has the distance you traveled during that one tenth of a second, or at least very close to it, because we are assuming it does not change much. Now all he has to do is sum up all those distances along the x, y and z directions and magically he knows where you finally arrived. Did you make it to work? He would know. It is a matter of just a lot of multiplications followed by a glorious summation of them. Meticulous and monotonous indeed, I mean really how many tenths of a second go by on your journey to work? But it gets the job done. If you stopped for gas he would know it. Having that map in hand, he would even know which one you stopped at and when you did it. Thus is the way of the calculus.

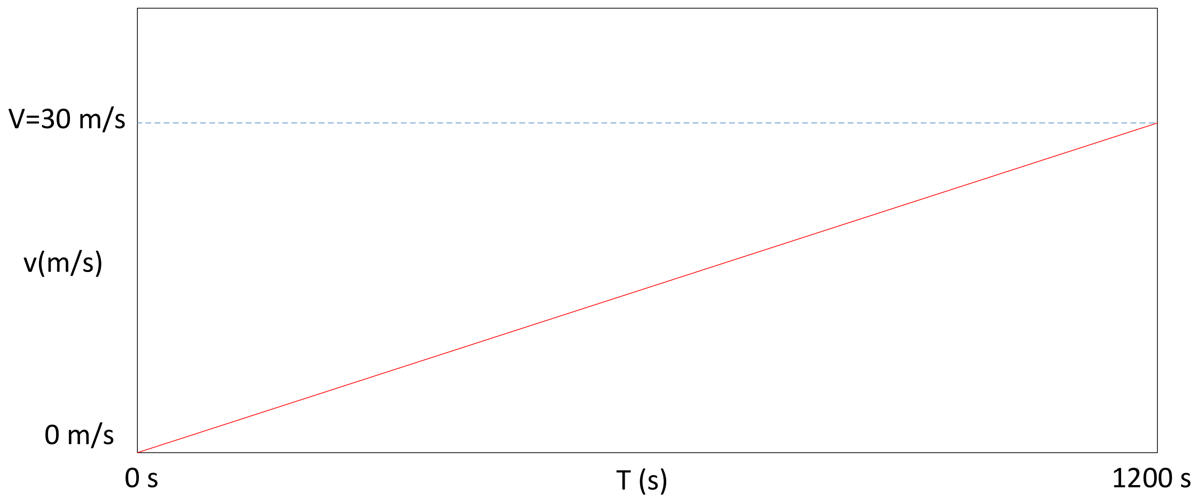
Now physics provides us a way to describe all that motion. But it has its limitations that need be discussed. Mathematics comes to the rescue of physics, at least in the sense of an approximation of the complex. The formula that follows describes a type of motion that we can relate to. Limited in multiple senses. It can only be applied to one of those directions we described. Need all three directions? You need apply the formula to the three possible directions of travel. Let that not bother us of the moment and just apply it to one of those directions. Instead of traveling to work. Replace the car with a train. Have the track running due north, have the train spend its first moving moments speeding up to some constant speed on its way to say Albuquerque. After reaching a constant speed then it just chugs along then at that speed and when it approaches Albuquerque and in order not to whiz right by it, the train begins to slow down coming to a final stop at the Albuquerque train station. If you're willing come aboard, boarding time is now and enjoy the ride. Here is the formula we will apply.

$$f(t)=\frac{1}{2}at^2+vt+c$$

The goal of this deceitfully feeble minded fellow is to describe the position of the train at any time 't'. That's f of t. That's how you read the left side of the equals sign. The function f(t) transforms 't' into a corresponding 'p'. 'p' being the position or how far we have traveled since we started moving. Thus we can apply the formula for any given 't' and out will pop the position of the train on its travel north to Albuquerque. Limitations abound in this world as they do here as well. The domain that is the region of time you may apply it assumes that 'a', 'v' and 'c' are known at the start of the domain and 'a' does not change during the duration of such domain. If we are at the start of such a domain, then 't' is zero, and the first two terms disappear into the midnight void leaving us c. That is our initial position. Any reference frame may be chosen for 'c'. We can choose one such that our train's initial position prior to moving is zero. For our story to continue, let's choose some units and graph out the sunny, sandy desert trip we are taking to Albuquerque. Yea, I know you all like miles per hour. But to make life simpler we will use meters per second. Meters for distance and seconds for time. If it bothers you just convert it as 1 MPH is nothing more than about (1/2 a meter per second). Here is our journey graphed out for us for easy manipulation except we plan to show our velocity on the graph's y axis, and time on the x axis:

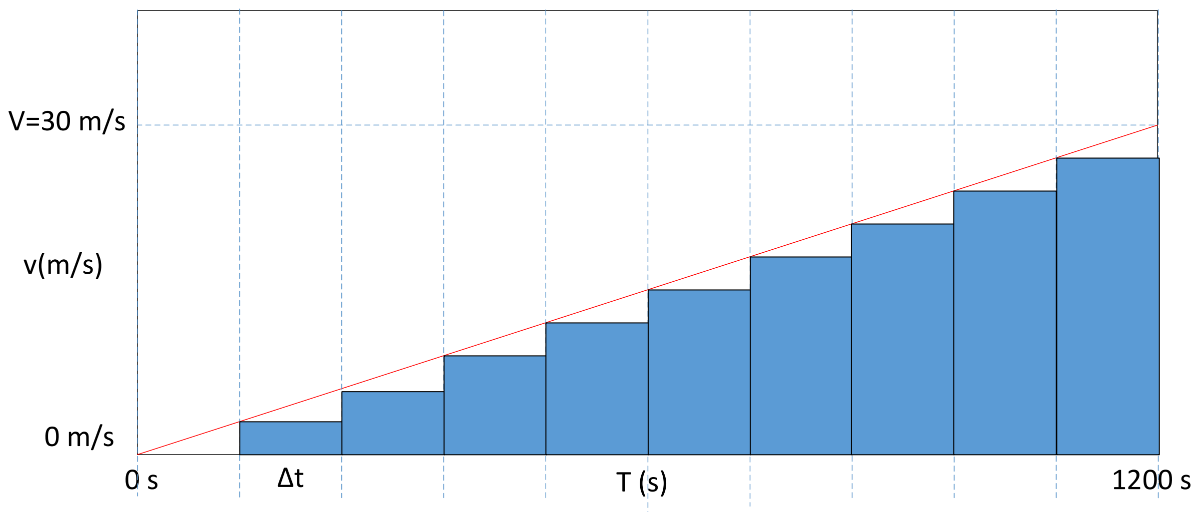


In words, we start at point zero, accelerate until we reach a speed of 30 m/s. Travel at constant speed for the next 7200-1200 or 6000 seconds, then proceed to slow down (decelerate). Until our speed is back to zero at 7800 seconds, where we reach Albuquerque train station. We have our speed here being plotted versus time. The formula can now be brought to bear to determine how far we have traveled to get to Albuquerque. But we just can't apply it to the whole trip we have to take it in pieces. If you were listening earlier we clearly stated your 'a' must remain constant during the domain and you must know your initial 'v' and 'c' for this to work out. This is true for three regions of our trip. From 0 to 1200 s, from 1200 s to 7200 s and from 7200 s to 7800 s. Please consider 'a'. What is it you may ask, it is how quickly your speed is changing. Look at our graph, just the first region from 0 s to 1200 s:



We want to know how quickly our speed is increasing. Well we can see from the graph we go from full stop to 30 m/s in 1200 seconds. What we want to know is how much does our speed increase with each passing second. We can see from the graph the rate (the slope of the line) is not changing it is a straight line. As such we see after 1200 seconds it is 30 m/s. So then what is our speed after the <u>first</u> second? It would be $1/1200^{th}$ of a movement along the t axis implying we just divide 30 m/s by 1200 seconds thus our speed is increasing by 30 m/s divided by 1200 s which is 0.025 m/s/s. In other words for each passing second our speed increases by 0.025 m/s. Notice also our initial position and velocity are both zero in this situation, so how far will we travel during this period? Looking at the formula; 'c' is zero, because our initial position is zero. 'v' is also zero as we are not yet even moving. At this point you should be on the train comfortably in your seat awaiting our departure. Because 'v' is zero so is vt because anything multiplied by zero is zero thus the only thing we need concern ourselves with is $½at^2$. That will tell us the distance we have moved over 0 to 1200 seconds. Applying the formula to we get $½(0.025)*(1200)^2$=18,000 meters. This is not the interesting part though, the question really arises as to why would taking one half of the acceleration and multiplying it by the square of the elapsed time give us the distance traveled?

Remember in the beginning of this article it was pointed out to you that if you knew the velocity at every instant along a duration you could then calculate the distance traveled by multiplying each interval known by the speed at the start of that interval, this would give you an approximation of the distance traveled. We can apply that here, let's break it up into say 10 parts and have a look at what happens:



So here we are we have broken up this fellow into 10 pieces each one forms a rectangle. If your train moved like this your ride would be a horrible mess of rapid changes in speed, best be buckled in with a 6 point race car seat belt in that case. But notice something interesting about it. Each rectangles width is Δt. And what is the height of the first rectangle there? Well Δt time has passed to the start of that fellow so the height would just be the slope of the line that is 'a' the acceleration multiplied by how long that slope has been rising like that and that is just 'a' times Δt or a*Δt or 0.025 m/s/s * 120 s = 3 m/s. That should not surprise us we reach 30 meters per second at the end of the graph, here we reach 3 m/s in one tenth of that. Which is simply one tenth of our final speed. Now how far did we move during this first rectangle span of Δt? If we assume we moved at that speed the entire time? Well it's just the 3 m/s * 120 s or 360 meters. But look at what happened in the math. We got 3 m/s by simply taking a* Δt and we got the distance by multiplying that by Δt. Or distance traveled = a* Δt* Δt or just '$a\Delta t^2$'. But look that is just the Area of the rectangle is it not? So if we take the height to the start of each rectangle we can then find the area under the rectangle, but that area is the distance traveled for that rectangle. Notice how that area is all under the red line of our true speed, then it stands to reason that the sum of the areas of the rectangles represents an approximation of our true total distance traveled. Let's see how close we are when we sum all 10 of these rectangles, we know we will be off because of those little triangles of error under each rectangle here is a table showing the summation:

RECTANGLE	TIME	SPEED	DISTANCE TRAVELED
1	0	0	0
2	120	3	360
3	240	6	720
4	360	9	1080
5	480	12	1440
6	600	15	1800
7	720	18	2160
8	840	21	2520
9	960	24	2880
10	1080	27	3240
		SUM:	16200

The formula stated we traveled 18,000 meters. Here our approximation has us at 16,200 meters. Were close but not that close. But what happens if I make Δt smaller? Say I break it into 100 parts? Laborious as it is here is the result (this time I will not show the table, as it is way too long to place in the document just the sum) 17,820. Well that is getting a lot closer to the correct answer of 18,000. What if we break it up again into 1000 rectangles what do we get (I am using Microsoft excel by the way to get these answers, the math is too laborious for me to do by hand!). Well with 1,000 rectangles we get:

17,982.

Well that is pretty close to 18,000.

But what happens if we go to 10,000 rectangles: (excel to the rescue please):

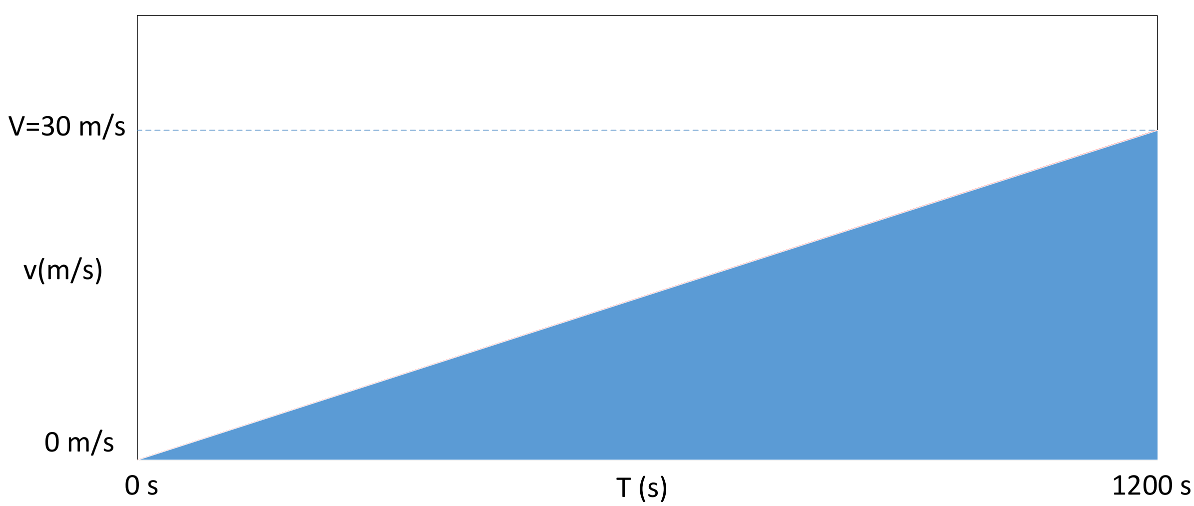
17,998.2 meters.

Now that is pretty darn close to 18,000. I mean say it was dollars, and I told you I would give you 18,000 dollars but only gave you 17,998 dollars and 20 cents. I short changed you by a dollar and 80 cents. Not too bad. Would you refuse the 18,000 dollars and state I was short changing you?

But what is all this insanity leading up to? Well look at what's happening in table form:

Number of Rectangles	Sum	Percent Error from 18000
10	16200	10.00%
100	17820	1.00%
1000	17982	0.10%
10000	17998.2	0.01%

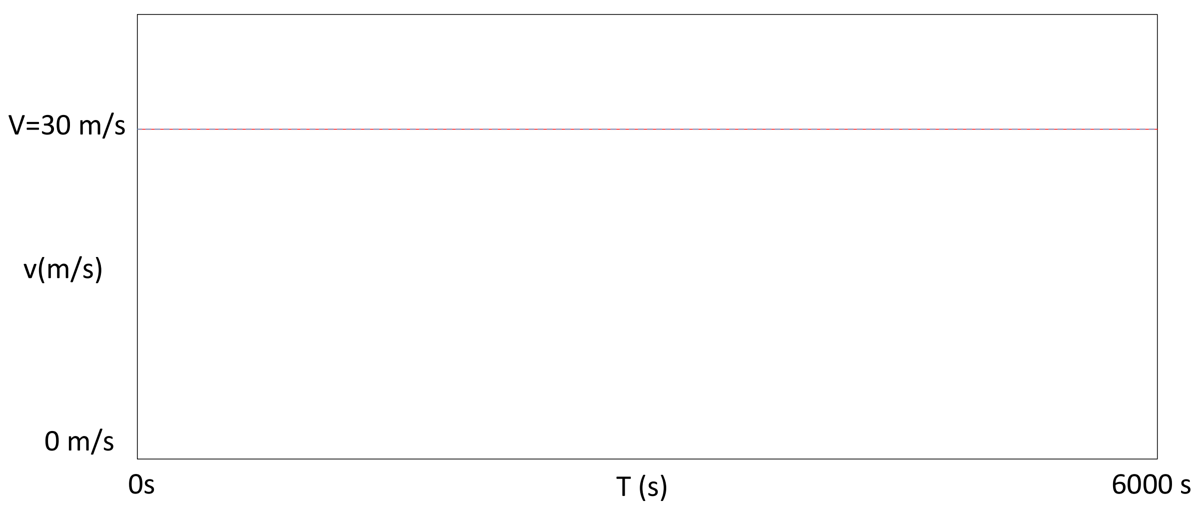
Notice a pattern? Each time we increase the number of rectangles by 10 the error from the true distance traveled decreases by an order of 10. In fact if we make the number of rectangles say 100,000 the error would be 0.001% and the number would be 17,999.82. Taking it 1 step further to 1 Million rectangles the error would be 0.0001% and the number would be 17,999.982. If I continue this process and make Δt extremely small, the graph fills in and looks like this:



That is you cannot see the rectangles anymore because they all blend into one another. Now think about this, what happens when you sum all those small Δt's together, do you not get the final number

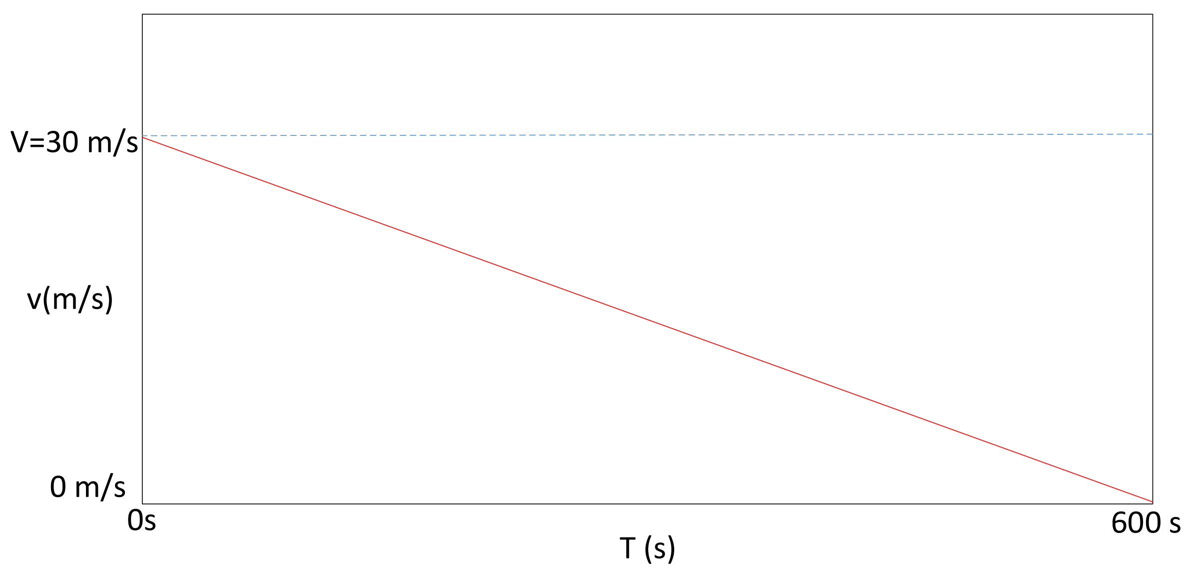
1200 s which is just ‘t’ in our case? Well you do! And those rectangles what where they? They were distances traveled if you take their area correct? Well here we did so to a fine degree and what does that area approach when we sum all those really thin rectangles? The area in this case under the line that represents our distance traveled. But now we can generalize to a solution that area is a triangle is it not? Just look at it. What is the area of a triangle is it not simply the base times the height divided by two. Like it’s half the rectangle when you take the speed up there on the vertical at 30 m/s and multiply it by the width which is just ‘t’. But that 30 m/s we got there using ‘at’ did we not? Then the distance traveled is that triangle with height ‘at’ multiplied by its width ‘t’ divided by two or $\frac{1}{2}at^2$. Thus the answer falls out and the proof is over my friend. We just proved by nothing but geometry and adding a huge number of really thin anorectic rectangles that the distance traveled when accelerating at constant ‘a’ is nothing more than $\frac{1}{2}at^2$. But that is what the formula states. The game is over on that term. So we traveled 18,000 meters during this interval. I hope you can see it, it is now so clear or should be that at constant acceleration the distance traveled due to that acceleration is just $\frac{1}{2}at^2$ and that should now be very obvious to you, lest my quest to provide you enlightenment has failed me. Let’s move on to the next domain in our problem.

The middle interval appears trivial, the velocity is constant. The value ‘a’ is zero. As the slope of the line reflects such value, and the slope flat thus zero. Your trip now at constant speed. We all intuitively enjoy such a movement, you can sip on a cup of coffee and need not worry about it sloshing around on you. The cup of coffee like you seems to have an implicit serenity about such things. With any slope in that velocity and things need buckled in and strapped down. Just like people, things, objects in the world just don’t like being pushed around and thus resist as Newton made clear to us with F=ma. ‘a’ again representing that change in speed that slope. Here ‘m’ is how much of that something there is, the more of something the more it resists and pushes back on any change. ‘F’ here is the force, the amount of talking back the object does about it. The greater the mass ‘a’ the greater talk back you get. Just like the most massive bully at your high school was usually the biggest big mouth as well. My digression now apparent to me, back to our trip. Here is the next interval. Not the most exciting graph, as it appears not much is happening, but looks are deceiving:



Notice our time has ‘reset’ itself. We were 1200 seconds into our trip when we got here. Why reset it you may ask? We used the slipping away ‘t’ up in that first interval and out popped its results for us: Final distance traveled 18,000 Meters, final speed at that point: 30 m/s. You cannot re-use it now. Like my wife tells me about money you can’t spend it more than once. It must reset to zero for the very reason that is gave us the start of our new interval. So here we subtracted out that 1200 seconds. Now reapplying our formula, the complex ½at^2 part now vanishes as 'a' is zero and we are left with vt + c. C being 18,000 meters, it not displayed on the graph, it was the outcome. 'v' now very clear and flat sitting at 30 m/s. Remember at the start of our journey the distance traveled simply turned out to just be the area under the curve? In this case our 'curve' is just a straight horizontal line, not just any line though a 'flat' line, a line with no slope, at 30 m/s and 't' here now being 6000 seconds. Here that area is just a rectangle with height of 30 m/s and length of 6000 seconds, being just the product of the two but that is just what the formula states, (vt). So via the math 30 m/s * 6000 s = 180,000 meters. I hope you note I missed something did I not? The formula implies 'c' must be taken and added from what we gained earlier and the distance traveled 180Km+18Km: 198Km grand total.

Now our trip takes its final turn. Brakes must be applied, lest will slip right on by Albuquerque train depot. The last part of our journey, time to start slowing down the train, it is large and heavy and we need not want to apply to much force and burn up the poor brakes: remember F=ma. Well trains have plenty of 'm', don't believe me just try to stop one. We are limited on how much 'F' our brakes can handle before they start slipping on the track, thus how much 'a' we can afford to invest. With 'm' being large then 'a' must be small to keep 'F' reasonable. Here I am digressing into Newtonian physics. But the article was all about physics in the first place. Velocity and Motion are physical things. By digressing into F=ma, I managed to show you where the 'a' is coming from in the first place. You need 'F' to create 'a' that is to get started to get moving that massive train, and a large engine to get to 30 m /s in that first 1200 seconds. That is we need a lot of 'F' to generate a little 'a' with all that massive 'm'. Same with stopping only the brakes involved now. 'a' takes on a different form in our formula implying a loss of velocity over a time and not a gain, and thus will be negative in value. But that makes sense, just look at the graph, is it now sloping downward as time moves forward. It is going downhill contrary to the start of our trip where we had an uphill journey. Per the aphorism: "Everything that goes up must come down" applies here as well. Here is the graph of the final leg of our trip:



Note the 'a' here is now sloping downward. Thus the 'a' term will be negative. So how do we calculate the slope in this situation? Compared to the start of our trip where 'a' was positive is now being negative. I will use a different technique, a different way of looking at it that is just as valid. The slope of a line is nothing more than how much movement to do you get along the y axis for a given amount of movement along the x axis. To calculate that you take the final speed minus the first speed and divide it by the final time minus the start time or: (0 m/s – 30 m/s) / (600 s – 0 s) = -0.05m/s/s or we slow down by 0.05 m/s for each passing second. Now in this situation we have an initial c of 198,000 meters and we now have an initial v here of 30 m/s. Now the formula come full to bear to give us an answer. The first term being (1/2)*-0.05t^2 the second: vt and the final 198,000. Carrying out the math: (1/2)-0.05(600*600) + (30*600) + 198,000 = -9000 + 18,000 + 198,000. Gives us the total distance to Albuquerque: 207,000 meters. We need not repeat the exercise for the area under that curve, we see it is just a triangle again. What may astound you here is look at the first term it gave us a <u>negative</u> distance? What is it telling us? Well look at our initial speed, it was 30 m/s. Had we continued our voyage at that speed for the next 600 seconds we would have traveled 30 m/s * 600 s or 18,000 meters. But is that not just the area formed by the rectangle on our graph of the dashed line at 30 m/s over that span of 600 seconds? Notice that happens to be our middle term, v being our initial velocity. Now 'a' comes to the rescue with a negative term subtracting out from what would have been that distance. But notice 'a' when applied $\frac{1}{2}at^2$ is just the area of the triangle formed below our sloping line. Because we are slowing down we subtract that out, and our forward distance traveled during this interval is simply 18,000 minus the 9,000 or 9,000 meters were traveled, add the distance already traveled at the start of this interval of 198,000 meters and we arrive at our total distance traveled for the entire trip of 207,000 meters. Putting that in unit's people prefer it is about 128 miles. And in case you are wondering that 30 m/s turns out to be around 67 miles per hour.

Where did we start? Here it is on the map:

So our train journey starting basically in the middle of nowhere in a barren desert just north of Alamogordo and Truth or Consequences bisecting the distance between the two and no there is no such train starting here nor ending. The whole story just to drive home the point and enlighten you on the Truth and Consequences of the most famous of motion formulas and how it really can be applied to the real world and give you a real understanding of where it really comes from anyways.

$$f(t)=\frac{1}{2}at^2+vt+c$$

I hope you found this article interesting. If you got nothing else out of it, please take home that space and time are inseparable. To talk of one without using or referring to the other is not possible except in the imaginations of the few. I would say they are simply two sides of the same coin. Call it space-time.

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