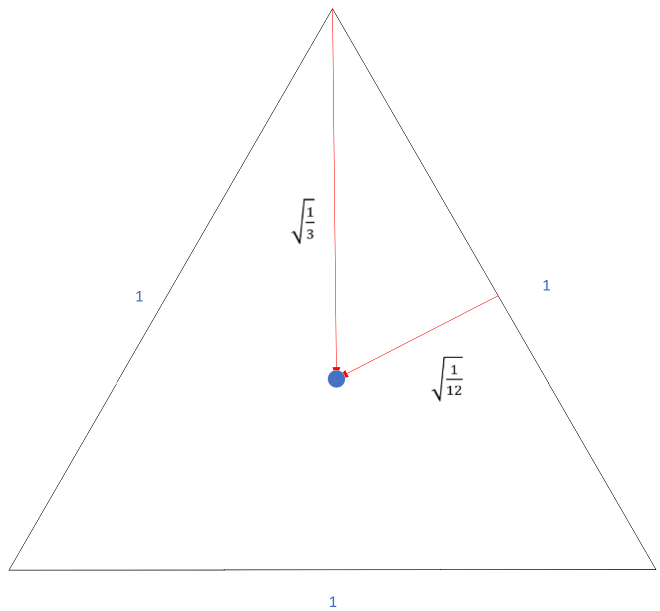
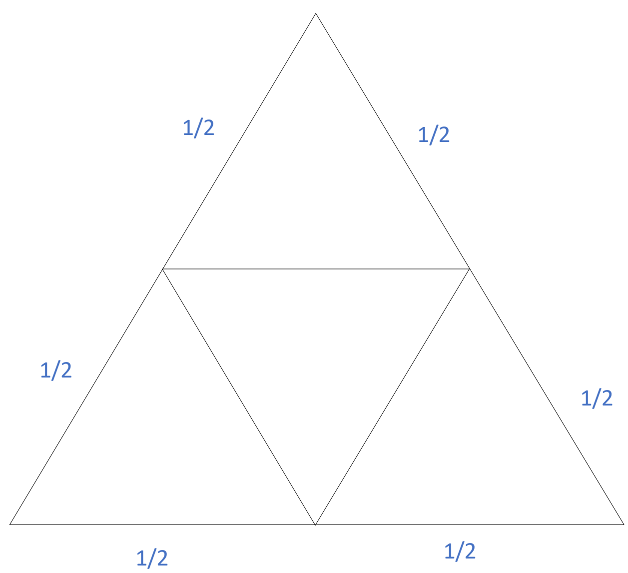
A TRIANGLE OF NO INSIGNIFICANCE PART 2

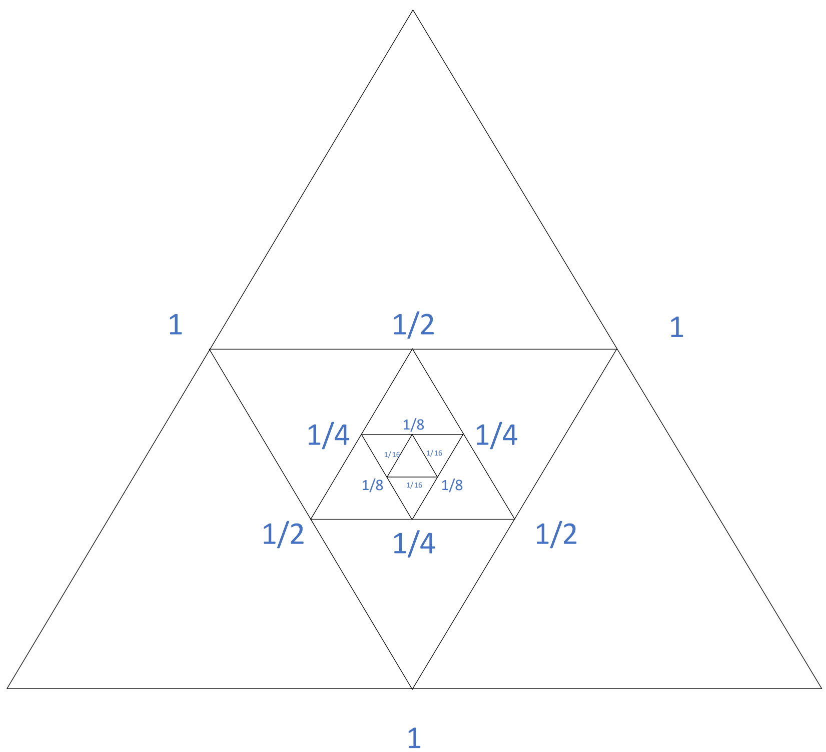
In the previous article, an equilateral triangle was constructed from 9 smaller equilateral triangles and Pythagoreans theorem was applied, to find the center of any equilateral triangle with unit length 1 for its sides. In retrospect to the previous article here was the solution we arrived at:



In this article we will arrive at the same solution but using a different method altogether. Like the previous solution, it still involves the geometry of equilateral triangles, but with a slightly different twist to it. Instead of constructing a larger equilateral triangle from 9 smaller ones, this solution will involve an infinite number of equilateral triangles that will converge to the solution. Consider the following:

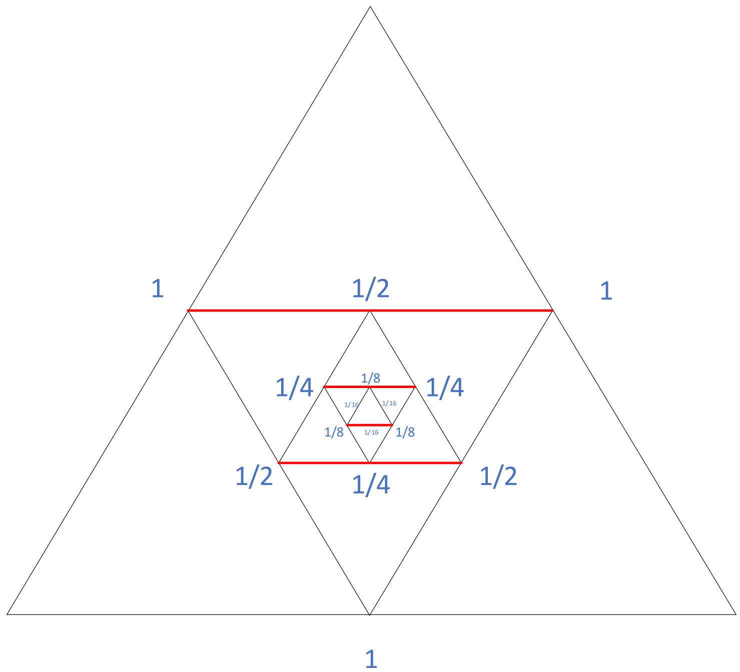


Imagine adding an upside down equilateral triangle that intersects the sides of the larger triangle, as shown here. Notice the intersections are at ½ of the unit triangle, therefore the inner triangle has sides equal to ½ of the outermost unit triangle. This process may be repeated indefinitely by swapping top for bottom of equilateral triangles and each successive one will have ½ the unit length of the previous one as shown here:



From this you get a sense that the middle triangle is converging towards the actual center of the outside unit length triangle in such a way that if we allow the process to continue indefinitely a small dot triangle would be sitting dead center of the outmost unit length triangle. It is this fact that allows us to calculate the center of the unit length triangle.

If we consider the following parallel lines (in red) of each triangle, then these lines are also converging on the center of the unit length triangle:



Using our earlier derivation for the height of a unit length triangle, we may ascertain the center of the triangle. If we consider downward pointing triangles as subtracting from that and upward pointing triangles as adding to that we get an equation that leads us to the center. Consider the first outer unit length triangle and the downward pointing triangle and we subtract the two, we get the distance to the largest red line with unit length 1/2 as such:

$$\sqrt{\frac{3}{4}}\left(1-\frac{1}{2}\right)$$

This expression denotes the distance from the top of the triangle to the red line with length ½. It is another way of saying, take the height $\sqrt{\frac{3}{4}}$ of the first triangle and subtract the height of the second triangle $\frac{1}{2}\sqrt{\frac{3}{4}}$. This process is repeatable, if we do the same operation again adding it to the sum we end up at the 1/8 red line distance:

$$\sqrt{\frac{3}{4}}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)\right]$$

Notice the pattern in the denominators are increasing by a factor of two for each expression such that 16 and 32 etc. would be next:

$$\sqrt{\frac{3}{4}}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{16}-\frac{1}{32}\right)+\left(\frac{1}{64}-\frac{1}{128}\right)\dots\right]$$

The triple dots implying the process may be repeated indefinitely. If we simplify this expression and perform the subtractions within then we get a simpler form as a summation:

$$\sqrt{\frac{3}{4}}\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}\dots\right)$$

The sequence (2, 8, 32, 128…) can be re-expressed as powers of 2 as all of them are powers of two or the sequence is the same as $(2^1+2^3+2^5+2^7\dots)$

Notice this sequence is having 2 raised to all the <u>odd</u> integers starting at 1, thus our original expression can be stated as:

$$c=\sqrt{\frac{3}{4}}\left(\frac{1}{2^1}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\frac{1}{2^9}+\frac{1}{2^{11}}+\frac{1}{2^{13}}\dots\right)$$

Here ***c*** represents the distance from the top of the unit length triangle to the actual center of the triangle, but this is an infinite series and can thus be mathematically expressed as follows:

$$c=\sqrt{\frac{3}{4}\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}}$$

This equation is the solution and now for the proof:

Ignoring for the moment the radical on the left we can find the sum of this infinite series:

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}}$$

Here is a table expressing the terms, we are interested in what it is converging on:

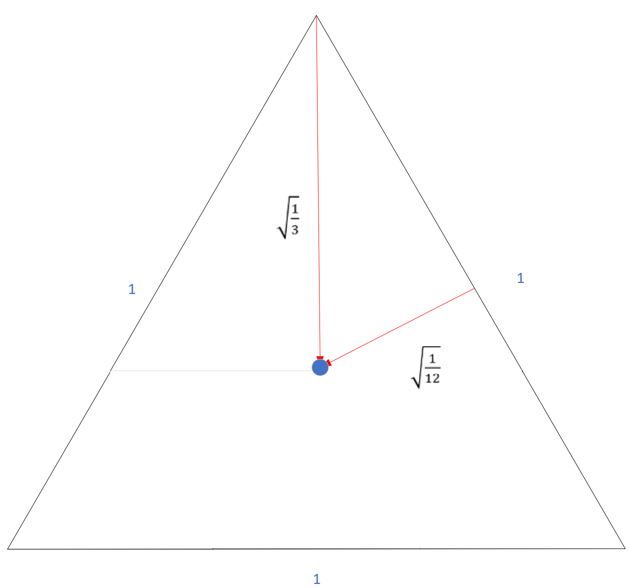
Term n	Value of Term n	Sum of terms
0	0.5	0.500000
1	0.125	0.625000
2	0.03125	0.656250
3	0.0078125	0.664063
4	0.001953125	0.666016
5	0.000488281	0.666504
6	0.00012207	0.666626
7	3.05176E-05	0.666656
8	7.62939E-06	0.666664
9	1.90735E-06	0.666666
10	4.76837E-07	0.666667
11	1.19209E-07	0.666667
12	2.98023E-08	0.666667
13	7.45058E-09	0.666667
14	1.86265E-09	0.666667
15	4.65661E-10	0.666667

After 16 terms it becomes clear this infinite sum is converging on 0.6666 or 2/3.

Thus, we can replace this limit that the infinite sum has converged upon into the equation and simplify:

$$c = \sqrt{\frac{3}{4}\left(\frac{2}{3}\right)} = \sqrt{\frac{12}{36}} = \sqrt{\frac{1}{3}}$$

This matches the answer we arrived at in the previous article:



Thus, the distance down from the pinnacle to the center matches. What about the other part? This distance from the center of the triangle to the center is 2/3 down the unit length triangle multiplied by the radical $\sqrt{\frac{3}{4}}$, and because of symmetry the distance from the center to the bottom of the triangle is the shortest distance to the edge and must be what remains or:

$$\sqrt{\frac{3}{4}}\left(1-\frac{2}{3}\right)=\sqrt{\frac{3}{4}}\left(\frac{1}{3}\right)=\sqrt{\frac{3}{36}}=\sqrt{\frac{1}{12}}$$

Thus, matching again, the previous articles solution. Infinite series are ubiquitous in the world of math, physics and geometry. Most of them not converging on anything but continuing to grow indefinitely while a minority stands out as the most important ones that converge on many mathematical constants such as the Euler's number 'e' which is used judiciously in physics and science. It is a convergence as follows:

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}$$

Many of these numbers show up in the real world in multifaceted ways and are the underpinning thing associated with the Calculus. Where the calculus takes an infinite series allowing the 'n' in our equations to shrink to zero and the sum to give way to infinity giving us the integral of something as long as the series converges. By the way e is not a rational number, it cannot be expressed as the ratio of two integers like our convergence upon 2/3.

Euler's number is non-repeating when expressed in decimal notation and goes on indefinitely but yet is still a limit, just like 2/3 but not expressible as a ratio of two integers here is the value of e:

e=2.71828182845904523536028747135266249775724709369995

This is expressed to 50 decimal places and the pattern will never repeat, there are some strange numbers out there like that one of them being the square root of 2 also being irrational.

I hope you enjoyed this article.

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