The Energy of a Swinging Pendulum

When most people think of pendulums, old dusty Grandfather Clocks come to mind, with the annoying bongs on the hour, dusty, fanciful majestic items composed of dark stained wood and metallic alloys, standing tall keeping time using a long shining brass rod with a dome circle on the bottom comes to mind. Barring where found, from an old home my mother in law lived in, to antique stores, these items now a thing of the past used a principle well known at the time: The period of swing is easily known and steady regardless of the angle of swing. Only the length of the pendulum was needed to determine the time to complete one swing. The amount of weight at the bottom has no bearing whatsoever on the period, as long as the attaching rod to it has a weight considerably less. This is the case in clocks using pendulums. The period being well know approximation as the following formula so long as the angle of swing is not beyond 20-30 degrees:

$$T \approx 2\pi\sqrt{\frac{L}{g}}$$

The typical Grandfather Clock has a swing of around 3-6 degrees maximum. This reduces friction in the clock thus increasing efficiency. The other goal is to reduce the swing period to the minimum, as each swing of the pendulum is effectively unwinding the clock. From the formula that means longer pendulums, thus the reason these clocks are so tall. For example, using the **kms** system we can solve for a given period, let's say you want the period to be 2 seconds, that is for the pendulum to swing out and back to the starting place will take 2 seconds, re-arranging the formula and solving for the length:

$$L = \frac{T^2 g}{4\pi^2}$$

With a period of 2 seconds this turns out to be:

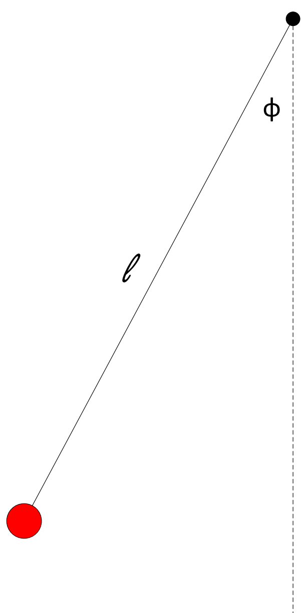
$$L = \frac{2^2(9.8)}{4\pi^2} = 0.9924\ Meters$$

Or about 3 feet and the reason Grandfather clocks are so tall, with the pendulum having length significant enough to warrant a tall clock.

All of this is well known. What if we want to know the velocity of the pendulum at all points during its swing? The clock provides energy to the pendulum to keep it swinging. If you look at the perspective of the weight that is swinging, during its path it has a mixture of potential energy and kinetic energy. Being the pendulum continues to swing at the same rate and angle, the total of those energies would be a constant or:

$$T_E = PE + KE$$

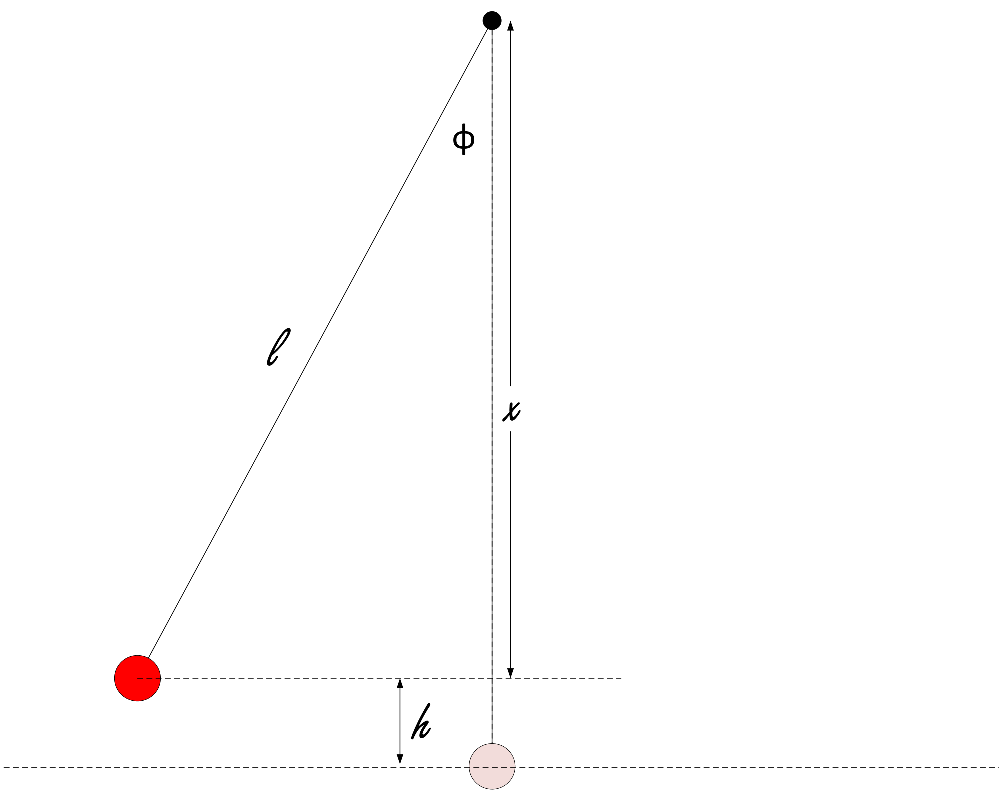
Let's consider a pendulum swinging at some angle ϕ with specified length:



At this position of motion, the red weight is not moving resulting in 0 kinetic energy and the potential energy is all that exists at this point. To derive that energy, we apply the formula:

$$PE = mgh$$

Here ***m*** is the mass of the weight, ***g*** is the gravitational constant and ***h*** is the height above the lowest point of motion. To calculate ***h*** consider the following:



The pink weight indicates the position of the weight at its lowest point. Note we have a right-angle triangle with ***ℓ*** being the hypotenuse and ***x*** being the adjacent to angle ϕ. Therefore, the length of ***x*** is:

$$\ell \cos(\emptyset)$$

But the length of the pendulum at this point is still ***ℓ*** therefore ***h*** must be what remains or:

$$h = \ell - \ell\cos(\emptyset) = \ell(1 - \cos(\emptyset))$$

But what is the potential energy at the lowest point? Is it not zero? If so and conservation of energy must be maintained, then at this point and at this point only, all of the energy is kinetic energy that was potential energy implying that they must be equal because no energy was gained or lost in a perfect pendulum therefore:

$$mg\,\ell(1-\cos(\emptyset)) = \frac{1}{2}mv^2$$

Here the mass is on both sides of the equation and can cancel:

$$g\,\ell(1-\cos(\emptyset)) = \frac{1}{2}v^2$$

Being that the maximum velocity is what we are seeking, it appears it is not dependent on the weight, but only the angle of swing and the length of the pendulum, solving for ***v*** we get:

$$v = \sqrt{2g\ell(1-\cos(\emptyset))}$$

Using our length for a 2 second swing of 0.9924 Meters we can solve for the speed in meters per second:

$$v = \sqrt{2(9.8)(0.9924)(1-0.9945)} = 0.327\ Meters\ per\ Second$$

Using the laws of conservation of energy, we have derived the formula for the maximum velocity of a swinging pendulum.

What about its velocity during the rest of the time? It just so happens that with respect to time the oscillating pendulums velocity is a sinusoidal pattern. We now know the maximum velocity, if plotted on a velocity vs time plot this would be the maximum amplitude of a sinusoid. We also know the period, that is the frequency of the pendulum. From this we can show the velocity as a function of time:

$$v(t) = v_{max}\sin\left(\frac{2\pi t}{T}\right), where\ the\ angle\ is\ in\ radians$$

Therefore, we know the velocity as a function of time. Notice the velocity can take on negative values, this indicates when the pendulum is swinging back to its starting position, while positive values indicate it is moving away from the starting position. The starting position is when the pendulum is at maximum height to the left.

Hope you enjoyed this article.

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